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Monodromy and Stability of a Class of Degenerate Planar Critical Points[1]

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Necessary and also sufficient monodromy conditions for a large class of degenerate singular points of planar differential systems are given. For these systems, we also find a computable expression of the principal term of the asymptotic expansion of the return map, which gives the stability of the point.

Key Words: degenerate critical point; monodromy problem; center focus problem; stability; blow-up procedure.

1. INTRODUCTION AND MAIN RESULTS

The study of the critical points for planar analytic differential equations, $\dot{x} = X(x)$, is almost totally solved. It is possible to know which is the

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behaviour of the solutions of a planar analytic differential equation in a neighbourhood of an isolated critical point p, in all cases except in the so-called *monodromic case*. Remember that this case is the one in which the solutions of the differential equation turn around the critical point.

The first problem appears when we have to decide whether a critical point is of monodromic type. This is usually done by the blow-up procedure. A more difficult problem is to give algorithms to know the stability of the monodromic critical points.

The case in which the differential matrix of the vector field at the critical point, $DX(p)$, has pure imaginary eigenvalues (different from zero) was already solved by Poincaré and Lyapunov.

The situation in which $DX(p)$ has all the eigenvalues zero, but it is not identically zero (usually called nilpotent case) has been solved by Moussu, see [1, 13].

The case in which $DX(p)$ is identically zero is much more difficult. If, after performing the first polar blow-up, there appear no critical points, then the theory developed by Poincaré and Lyapunov can be reproduced. In the general monodromic situation, it is known that the return map Π around p can be expressed as $\Pi(x) = V_1 x + o(x)$, for some nonzero constant V_1. Medvedeva in [11] gives a procedure to compute V_1 for any monodromic singular point p. To apply Medvedeva's result it is necessary to perform all the blow-ups to desingularize the point, in order to decide if it is monodromic, and to compute V_1. This result, as far as we know, is the last of a series of papers about this subject [4, 9, 10, 11]. Observe that V_1—which in the case of a critical point with pure imaginary eigenvalues corresponds with exp $(\operatorname{div} X(p))$—decides the stability of the critical point just when it is not 1. The stability of p when $V_1 = 1$ is yet an open problem. See [12] for some results in this direction.

To describe our contribution to the monodromy and stability problems, we fix some notation.

Let Φ be the set of germs of planar real analytic vector fields at the origin. For $X \in \Phi$, we denote by X^1 and X^2 its components. We also denote by X_k or X_k^1 and X_k^2 the corresponding homogeneous components of degree k. For $k \geqslant 2$, set

$$\Phi_k = \{X \in \Phi : X_0 = X_1 = \cdots = X_{k-1} = 0 \quad \text{and} \quad X_k \neq 0\}.$$

We associate to each $(X^1, X^2) \in \Phi_k$ the differential system

$$\begin{aligned} \dot{x} &= X^1(x, y), \\ \dot{y} &= X^2(x, y). \end{aligned} \tag{1}$$

For $X \in \Phi_k$ and for $i \geqslant k$, let

$$F_i(X)(\theta) = \cos(\theta)X_i^2(\cos(\theta), \sin(\theta)) - \sin(\theta)X_i^1(\cos(\theta), \sin(\theta)),$$
$$R_i(X)(\theta) = \cos(\theta)X_i^1(\cos(\theta), \sin(\theta)) + \sin(\theta)X_i^2(\cos(\theta), \sin(\theta)).$$

Any zero of $F_k(X)$ is called a characteristic direction of X. As we will see below (see Lemma 5) if $X \in \Phi_k$ is monodromic then k is odd, the number of characteristic directions of X in $[0, \pi)$ is bounded above by $\frac{k+1}{2}$ and $F_k(X)(\theta)$ is not identically zero and does not change the sign. Without loss of generality, we fix our attention in the case $F_k(X)(\theta) \geqslant 0$. Note that this last condition implies that any characteristic direction has even multiplicity. Let $l \leqslant \frac{k+1}{2}$ and $\omega = \{\theta_1, \ldots, \theta_l\}$ with $\theta_i \in [0, \pi)$ such that $\theta_i \neq \theta_j$ if $i \neq j$. We denote by $\Phi_{k,\omega}$ the set of vector fields in Φ_k having $\omega \cup \{\omega + \pi\}$ as a set of characteristic directions and verifying that $F_k(X) \geqslant 0$.

Given a function f, continuous in $[0, 2\pi] \setminus \{\theta_1, \theta_2, \ldots, \theta_k\}$ we define the *Cauchy Global Principal Value* (GPV) of $\int_0^{2\pi} f(\theta)\, d\theta$ as the following limit (if it exists):

$$\text{GPV} \quad \left\{ \int_0^{2\pi} f(\theta)\, d\theta \right\} := \lim_{\varepsilon \to 0} \int_{I_\varepsilon} f(\theta)\, d\theta,$$

where

$$I_\varepsilon = [0, 2\pi] \setminus \bigcup_{i=1}^{l} (\theta_i - \varepsilon, \theta_i + \varepsilon).$$

Our main results will be precisely stated in Theorems 1 and 2, but firstly we prefer to describe them.

- We will describe a "generic" subset inside the set of monodromic vector fields in $\Phi_{k,\omega}$, defined by simple, semi-algebraic and testable conditions.
- For "most" monodromic vector fields in $\Phi_{k,\omega}$, the stability of p is given by the sign of

$$\text{GPV} \quad \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\, d\theta.$$

- As can be seen from the papers [6, 11], there are monodromic vector fields ("residual") for which the above global principal value does not give the stability of the critical point.

To be more precise, we define some subsets of $\Phi_{k,\omega}$ which can be thought as approximations to the set of monodromic vector fields at the origin. Let

$S_{k,\omega}$ be the set of vector fields in $\Phi_{k,\omega}$ verifying the following properties for all $\theta_j \in \omega$:

(a) $R_k(X)(\theta_j) = 0$,
(b) $F_{k+1}(X)(\theta_j) = 0$,
(c) $[(F_k(X)'' - 2R_k(X)')F_k(X)''](\theta_j) > 0$,
(d) $[(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)](\theta_j) < 0$.

As we will see, the vector fields in $S_{k,\omega}$ are the vector fields verifying that each characteristic direction can be desingularized applying two (polar or directional) blow-ups unfolding the origin into two hyperbolic saddles (see [2, 3] or [5] for more references about the desingularization process).

Now let $N_{k,\omega}$ be the set of vector fields in $\Phi_{k,\omega}$ verifying the following properties for all $\theta_j \in \omega$:

(a) $R_k(X)(\theta_j) = 0$,
(b) $F_{k+1}(X)(\theta_j) = 0$,
(e) $[(F_k(X)'' - 2R_k(X)')F_k(X)''](\theta_j) \geqslant 0$,
(f) $[(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)](\theta_j) \leqslant 0$,
(g) $F_{k+2}(X)(\theta_j) \geqslant 0$.

Clearly, $S_{k,\omega} \subset N_{k,\omega}$. Lastly, set

$$M_{k,\omega} = \{X \in \Phi_{k,\omega}\colon \text{ the origin is monodromic and } F_k(X) \geqslant 0\}.$$

Our main results are summarized in the following two theorems.

THEOREM 1. *Following the above definitions, the next inclusions hold*

$$S_{k,\omega} \subset M_{k,\omega} \subset N_{k,\omega}.$$

Moreover, the following properties are satisfied:

(1) *If* $X \in S_{k,\omega}$ *and* $Y \in M_{k,\omega}$ *then* $X + \varepsilon Y \in S_{k,\omega}$ *for all* ε *small enough.*

(2) *Any* $X \in M_{k,\omega}$ *can be approximated, uniformly over compacts, by a sequence* $\{X_n\}_{n\in\mathbb{N}}$, with $X_n \in S_{k,\omega}$.

THEOREM 2. *Let* $X \in S_{k,\omega}$. *Then the following statements hold*:

(i) GPV $\int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta$ *exists.*
(ii) *The return map associated with the origin of* (1) *has the form*

$$\Pi(x) = V_1 x + o(x),$$

where

$$V_1 = \exp\left\{ \text{GPV} \quad \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta \right\}. \tag{2}$$

Remark 3. Theorem 1 is a genericity result. Let U be a bounded, open neighbourhood of the origin. Consider $S_{k,\omega}(U) = S_{k,\omega} \cap \{\mathscr{H}(U)\}$, where $\mathscr{H}(U)$ stands for the set of analytic vector fields in U, and continuous in $\bar{U}$, endowed with the uniform convergence topology. Analogously define $M_{k,\omega}(U)$.

With the above topology the maps c_j and d_j which appear in the description of $S_{k,\omega}$ and defined from $M_{k,\omega}(U)$ to $\mathbb{R}$ given by

$$c_j(X) := [(F_k(X)'' - 2R_k(X)')F_k(X)''](\theta_j),$$

$$d_j(X) := [(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)](\theta_j)$$

are continuous. This result implies that $S_{k,\omega}(U)$ is open in $M_{k,\omega}(U)$. On the other hand, condition (ii) of Theorem 1 states that $S_{k,\omega}(U)$ is dense in $M_{k,\omega}(U)$.

In fact, by considering in the set of real analytic germs Φ the usual topology given by the inductive limit of the topological spaces $\mathscr{H}(U)$ (see [14, pp. 54–60]) it is not difficult to see from Theorem 1 that $S_{k,\omega}$ is open and dense in $M_{k,\omega}$.

Remark 4. The case $F_k(X)(\theta) \leqslant 0$ can be reduced to the case $F_k(X)(\theta) \geqslant 0$ just by changing the sign of the independent variable in Eq. (1). In this new case, Theorem 2 works with

$$V_1 = \exp\left\{ -\text{GPV} \quad \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta \right\},$$

instead of the original formula for V_1.

The number V_1 is called the *generalized first Lyapunov constant* for system (1). Note that if there are no characteristic directions, then

$$V_1 = \exp\left\{ \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta \right\}.$$

This paper is organized as follows. In Section 2, we give the proof of Theorem 1 while in Section 3 the proof of Theorem 2 is displayed. The paper is ended with Section 4 in which we wonder about the following two problems, concerned with the usefulness of Theorems 1 and 2:

- When is it possible to check if a concrete vector field is an element of $S_{k,\omega}$?

- Given a vector field $X \in S_{k,\omega}$, when is it possible to compute

$$\mathrm{GPV} \quad \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\,d\theta?$$

The main result of Section 4, Theorem 13, is stated in Section 4.2. Finally, in Section 4.3, concrete examples of application are given.

2. PROOF OF THEOREM 1

We start stating the following well-known result.

LEMMA 5. *Let $X \in \Phi_k$ and assume that X is monodromic. Then the following assertions hold*:

(i) *$F_k(X)$ is not identically zero*,

(ii) *$F_k(X)(\theta)$ does not change the sign*,

(iii) *k is odd*,

(iv) *There are at most $\frac{k+1}{2}$ characteristic directions in $[0, \pi)$ and each characteristic direction has even multiplicity.*

Proof of Theorem 1. First we show that $M_{k,\omega} \subset N_{k,\omega}$. To prove this inclusion, let $X \in M_{k,\omega}$ and let $\theta_j \in \omega$. Consider the expression of (1) in polar coordinates,

$$\dot{r} = \sum_{i=k}^{\infty} R_i(X)(\theta) r^{i-k+1},$$
$$\dot{\theta} = \sum_{i=k}^{\infty} F_i(X)(\theta) r^{i-k}.$$

Note that $(0, \theta_j)$ is a critical point of the above system. Its linear part is given by

$$\begin{pmatrix} R_k(X)(\theta_j) & 0 \\ F_{k+1}(X)(\theta_j) & 0 \end{pmatrix}.$$

Hence $R_k(X)(\theta_j)$ must be zero. Otherwise $(0, \theta_j)$ would be a critical point with a nonzero eigenvalue and this fact would contradict the monodromy of the origin. On the other hand, also by the monodromy of the origin it follows that for r small enough $\dot{\theta}(r, \theta_j)$ and $\dot{\theta}(r, \theta_j + \pi)$ have the same sign. If $F_{k+1}(X)(\theta_j) \neq 0$, then these signs are the signs of $F_{k+1}(X)(\theta_j)$ and $F_{k+1}(X)(\theta_j + \pi)$. This last assertion is in contradiction with the fact that $F_{k+1}(\theta)$ is a trigonometric polynomial with odd degree. Therefore, $F_{k+1}(X)$ $(\theta_j) = 0$. Hence X satisfies (a) and (b).

In order to prove that X verifies (e)–(g) at θ_j we need to explicit the blow-up process. To do this, first we conjugate X by means of a rotation in such a way the characteristic direction θ_j is transformed into the direction $\theta = 0$. We denote by Y the vector field obtained from this conjugation. Some computations give the following relations:

(A) $Y^1(x,y) = \cos(\theta_j)X^1(M_{\theta_j}(x,y)) + \sin(\theta_j)X^2(M_{\theta_j}(x,y))$ and $Y^2(x,y) = -\sin(\theta_j)X^1(M_{\theta_j}(x,y)) + \cos(\theta_j)X^2(M_{\theta_j}(x,y))$, where

$$M_{\theta_j} = \begin{pmatrix} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{pmatrix}.$$

(B) For $i \geqslant k$, $F_i(Y)(\theta) = F_i(X)(\theta + \theta_j)$ and $R_i(Y)(\theta) = R_i(X)(\theta + \theta_j)$.

Next, we consider the following blow-ups:

(i) $(u,z) = (x^2/y, y/x)$. This blow-up is obtained by the composition of the two blow-ups $(x,z) = (x, y/x)$ and $(u,z) = (x/z, z)$.

(ii) $(x,w) = (x, y/x^2)$. As above, this blow-up is obtained by the composition of the two blow-ups $(x,z) = (x, y/x)$ and $(x,w) = (x, z/x)$.

Rewriting the new differential equations in terms of the original vector field, we get

$$\begin{aligned} \dot{u} &= \left(R_k(X)'(\theta_j) - \frac{F_k(X)''(\theta_j)}{2} \right) u + \mathcal{O}_2(z,u), \\ \dot{z} &= \left(\frac{F_k(X)''(\theta_j)}{2} \right) z + \mathcal{O}_2(z,u) \end{aligned} \tag{3}$$

and

$$\dot{x} = Y_k^1(1, wx) + \frac{\sum_{i>k}^{\infty} Y_i^1(x, wx^2)}{x^k} = p(x,w),$$

$$\begin{aligned} \dot{w} = {} & F_{k+2}(X)(\theta_j) + (F_{k+1}(X)' - R_{k+1}(X))(\theta_j) w + \left(\frac{F_k(X)''}{2} - R_k(X)' \right)(\theta_j) w^2 \\ & + xW(x,w) = q(x,w). \end{aligned} \tag{4}$$

From (3), $\left(\frac{F_k(X)''(\theta_j)}{2} - R_k(X)'(\theta_j)\right) F_k(X)''(\theta_j)$ must be greater or equal to 0. Otherwise, the corresponding critical point in the blow-up is a node and this contradicts the monodromy of X. Thus X must satisfy (e).

Note also that $\{x = 0\}$ is an invariant straight line of system (4). If

$$[(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)](\theta_j) > 0,$$

then a study of the power series expansion of the above expressions gives that there are two simple critical points in $x = 0$. This gives a contradiction with the monodromy of X. Hence X has to verify (f).

Lastly, if $F_{k+2}(X)(\theta_j) < 0$ then the system at $x = 0$ turns in opposite direction to the monodromic one. So (g) is proved. Thus, we have proved that $M_{k,\omega} \subset N_{k,\omega}$.

Now we prove that $S_{k,\omega} \subset M_{k,\omega}$. Let $X \in S_{k,\omega}$. Note that conditions (a)–(d) ensure that, after the above blow-up process, each singularity of the system associated with X has been desingularized obtaining a monodromic polycycle whose corners are hyperbolic saddles. So $X \in M_{k,\omega}$.

Next, we begin the proof of statements (1) and (2).

(1) Let $X \in S_{k,\omega}$ and $Y \in M_{k,\omega}$. Since R and F are linear operators we get that for all $i \geqslant k$, $R_i(X + \varepsilon Y)(\theta) = (R_i(X) + \varepsilon R_i(Y))(\theta)$ and $F_i(X + \varepsilon Y)(\theta) = (F_i(X) + \varepsilon F_i(Y))(\theta)$. Then since (a) and (b) hold for X and Y then $X + \varepsilon Y$ again verifies (a) and (b). Since X verifies (e) and (f) with strict inequality and Y satisfies (e) and (f), it follows that for ε small enough $X + \varepsilon Y$ verifies (e) and (f) with strict inequality, that is $X + \varepsilon Y$ satisfies (c) and (d). Therefore, we have proved that $X + \varepsilon Y \in S_{k,\omega}$ as desired.

(2) Without loss of generality, we can assume that $\pi/2 \notin \omega$. Let $X \in M_{k,\omega}$. We have already proved that $X \in N_{k,\omega}$. For each $\theta_i \in \omega$, set $z_i = \tan(\theta_i)$.

If $X \in S_{k,\omega}$ there is nothing to prove. Hence, we assume that $X \notin S_{k,\omega}$. Therefore for some $\theta_j \in \omega$ either (e), or (f) or (g) holds with an equality instead of a strict inequality. Consider $X_{\varepsilon,\delta} = X + \varepsilon Y + \delta Z$, where Y is any homogeneous vector field of degree $k + 1$ and $Z(x, y) = (x^2 + y^2)^{(k+1)/2}(-y, x)$. It is easy to see that $F_{k+2}(X_{\varepsilon,\delta})(\theta_j) = F_{k+2}(X)(\theta_j) + \delta$ and so choosing δ positive and small enough $X_{\varepsilon,\delta}$ satisfies (g) with strict inequality. The vector field Y has to be chosen in order to transform (e) and (f) into strict in-equalities. When Card $(\omega) = l < \frac{k+1}{2}$ this is very easy to be done. It suffices to take

$$Y(x, y) = (x^2 + y^2)^{\frac{k-1-2l}{2}} \prod_{i=1}^{l} (y - xz_i)^2(-y, x)$$

and ε is positive and small enough. Note that the above vector field is homogeneous of degree k because $l < (k + 1)/2$. When $l = (k + 1)/2$ it is no more polynomial. In this latter case the construction of Y is more involved. We will work with polynomials instead of trigonometrical polynomials.

Consider the following two polynomials:

$$G(z) = X_k^2(1,z) - zX_k^1(1,z) \qquad \text{and} \qquad S(z) = X_k^1(1,z) + zX_k^2(1,z).$$

Easy computations show that the following statements hold:

(i) For $\theta \neq \pi/2$, $G(\tan(\theta)) = \frac{1}{\cos^{k+1}(\theta)} F_k(X)(\theta)$ and $S(\tan(\theta)) = \frac{1}{\cos^{k+1}(\theta)} R_k(X)(\theta)$.

(ii) The degree of $G(z)$ is exactly $k+1$.

(iii) $\theta \in \omega$ if and only if $\tan(\theta)$ is a root of $G(z)$. Moreover for $i = 1, \dots l$, $S(z_i) = 0$.

(iv) $(S - zG)(z) = (1+z^2)X_k^1(1,z)$ and $(zS + G)(z) = (1+z^2)X_k^2(1,z)$.

(v) For $i = 1, \dots, l$, $F_k(X)''(\theta_i) = G''(z_i)(z_i^2+1)^{-(k-3)/2}$ and $R_k(X)'(\theta_i) = S'(z_i)(z_i^2+1)^{-(k-1)/2}$.

In our situation, $G(z)$ is a polynomial of degree $k+1$ with $l = \frac{k+1}{2}$ roots of multiplicity two. Without loss of generality, we suppose that

$$G(z) = \prod_{i=1}^{l} (z - z_i)^2.$$

Then by (v),

$$F_k(X)''(\theta_j) = 2 \prod_{i=1,\ i \neq j}^{l} (z_j - z_i)^2 (z_j^2 + 1)^{-\frac{k-3}{2}} > 0.$$

On the other hand by (iii), there exists a polynomial $s(z)$ of degree at most $l-1$, such that $X_k^1(1,z) = \prod_{i=1}^{l}(z - z_i)s(z)$. By using (iv),

$$S(z) = \prod_{i=1}^{l}(z - z_i)\left[z \prod_{i=1}^{l}(z - z_i) + (1+z^2)s(z) \right]. \tag{5}$$

If X does not verify (c) then $F_k(X)''(\theta) = 2R_k(X)'(\theta)$ for some $\theta \in \omega$, that is

$$S'(z_j) = \prod_{i=1,\ i \neq j}^{l} (z_j - z_i)^2 (1 + z_j^2) \tag{6}$$

for some $j \in \{1, \dots, l\}$. Let us prove by contradiction that it is impossible that (6) holds for all $j \in \{1, \dots, l\}$. Assume that (6) holds for all $j \in \{1, \dots, l\}$. Then, by (5) we see that

$$S'(z_j) = \prod_{i=1,\ i \neq j}^{l} (z_j - z_i)(1 + z_j)^2 s(z_j).$$

Hence, using (6) we obtain

$$s(z_j) = \prod_{i=1,\ i\neq j}^{l} (z_j - z_i) = p'(z_j)$$

for all $j = 1, \ldots, l$, where $p(z) := \prod_{i=1}^{l}(z - z_i)$. Therefore $s(z) = p'(z)$. Note that $s(z) = lz^{l-1} + \cdots$. By substituting its expression into (5) we get that $S(z)$ has degree larger than $k+1$, which gives a contradiction. Therefore, if we denote by K the set of z_i's verifying (6), the above reasoning implies that the cardinality of K is at most $l-1$. Let $r(z)$ be a polynomial of degree $l-1$ such that for any $z \in K$, $\prod_{i=1,\ z_i\neq z}^{l}(z - z_i)(1+z^2)r(z) < 0$. This polynomial exists, just because the cardinality of K is less than l. Set Y the vector field defined as follows:

$$Y(x, y) = -x^k \prod_{i=1}^{l}\left(\frac{y}{x} - z_i\right)\left(r\left(\frac{y}{x}\right), \prod_{i=1}^{l}\left(\frac{y}{x} - z_i\right) + \frac{y}{x} r\left(\frac{y}{x}\right)\right).$$

Tedious computations show that for all $\varepsilon > 0$, and $\delta > 0$, the following statements hold:

(I) $F_k(X_{\varepsilon,\delta})''(\theta) = (1+\varepsilon)F_k(X)''(\theta)$ for all $\theta \in \omega$.

(II) $R_k(X_{\varepsilon,\delta})'(\theta) < R_k(X)'(\theta)$ for all $\theta \in K$.

Hence for $\varepsilon > 0$ and $\delta > 0$ we have constructed $X_{\varepsilon,\delta} \in S_{k,\omega}$. Observe also that $X - X_{\varepsilon,\delta}$ is a polynomial which tends to zero uniformly over compacts when ε, δ tends to zero. This ends the proof of the theorem. ∎

3. PROOF OF THEOREM 2

Before proving Theorem 2, we need some preliminary results. First, for the sake of completeness, we restate Lemma 8 of [6].

LEMMA 6. *Consider system*

$$\begin{aligned} \dot{x} &= -x(\alpha + f(x, y)) = P(x, y), \\ \dot{y} &= y(\beta + g(x, y)) = Q(x, y), \end{aligned} \tag{7}$$

where f and g begin with first-order terms, and, α and β are positive. Let $\sigma_{\varepsilon,\delta}(y)$ be the transition map of the flow from $\{x = \varepsilon\}$ to $\{y = \delta\}$ being ε and δ small enough and positive, then

$$\sigma_{\varepsilon,\delta}(y) = A(\varepsilon, \delta)y^{\alpha/\beta} + o(y^{\alpha/\beta}) \qquad \text{with} \qquad A(\varepsilon, \delta) = \frac{\varepsilon \quad \exp\{F(\delta)\}}{\delta^{\alpha/\beta}\exp\left\{\frac{\alpha}{\beta}G(\varepsilon)\right\}},$$

where

$$\lim_{\delta\to 0} F(\delta) = \lim_{\varepsilon\to 0} G(\varepsilon) = 0.$$

Next lemma studies the transition map near a characteristic direction. We refer to [8] for the definition of a semi-regular map. Given a continuous map $f : \mathbf{R} \to \mathbf{R}$ we denote as PV $\int_{-\infty}^{\infty} f(x)\,dx$, its principal value, defined as the following limit (if it exists):

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,dx := \lim_{y\to\infty} \int_{-y}^{y} f(x)\,dx.$$

LEMMA 7. *Let* $X \in S_{k,\omega}$, $\theta_0 \in \omega$. *For* $\bar{\varepsilon} > 0$ *small enough, set* $\varepsilon = \tan(\bar{\varepsilon})$, *and* $\mathscr{S}_+^0 = \{\tan(\theta_0 - \bar{\varepsilon})x \leqslant y \leqslant \tan(\theta_0 + \bar{\varepsilon})x\}$, *and* $\mathscr{S}_-^0 = \{\tan(\theta_0 + \pi + \bar{\varepsilon})x \leqslant y \leqslant \tan(\theta_0 + \pi - \bar{\varepsilon})x\}$. Let $\Delta_{\theta_0}^{\bar{\varepsilon}}$ *and* $\Delta_{\theta_0+\pi}^{\bar{\varepsilon}}$ *be the transition maps of the flow in* $\mathscr{S}_+^0$ *and* $\mathscr{S}_-^0$. *Then* $\Delta_{\theta_0}^{\bar{\varepsilon}}$ *and* $\Delta_{\theta_0+\pi}^{\bar{\varepsilon}}$ *are semi-regular maps,*

$$\Delta_{\theta_0}^{\bar{\varepsilon}}(r) = D_{\theta_0}^{\bar{\varepsilon}} r + o(r), \quad \Delta_{\theta_0+\pi}^{\bar{\varepsilon}}(r) = D_{\theta_0+\pi}^{\bar{\varepsilon}} r + o(r),$$

where $r^2 = x^2 + y^2$, *and*

$$D_{\theta_0+(\pi\pm\pi)/2}^{\bar{\varepsilon}} = \exp\left\{ H(\varepsilon) \mp \lambda \ \mathrm{PV} \int_{-\infty}^{\infty} \left(\left(\frac{\partial}{\partial x} \frac{p(x,w)}{q(x,w)} \right)\Bigg|_{\{x=0\}} \right) dw \right\}$$

with

$$\lim_{\varepsilon\to 0} H(\varepsilon) = 0, \qquad \lambda = \frac{F_k(X)''(\theta_0) - 2R_k(X)'(\theta_0)}{F_k(X)''(\theta_0)}$$

and p *and* q *defined in* (4).

Proof. Without loss of generality, we assume that $\theta_0 = 0$. Since $X \in S_{k,\omega}$ the blow-up process described by Eqs. (3) and (4), and studied in the proof of Theorem 1, desingularizes the direction $\theta = 0$ in two saddles, p_1 and p_2 (see Fig. 1), with hyperbolic ratios $1/\lambda$ and λ, respectively.

For system (3), the origin is a hyperbolic saddle for which the first quadrant corresponds to the hyperbolic sector of p_2 which comes from $\mathscr{S}_+$, and the fourth quadrant corresponds to the hyperbolic sector of p_1 also coming from $\mathscr{S}_+$ (see Fig. 2). The second and the third quadrant come from $\mathscr{S}_-$.

Now we want to describe $\Delta_0^{\bar{\varepsilon}}$ the transition map of the flow between $\Sigma_1 = \{y = -\varepsilon x\}$ and $\Sigma_4 = \{y = \varepsilon x\}$. The description of $\Delta_\pi^{\bar{\varepsilon}}$ is analogous.

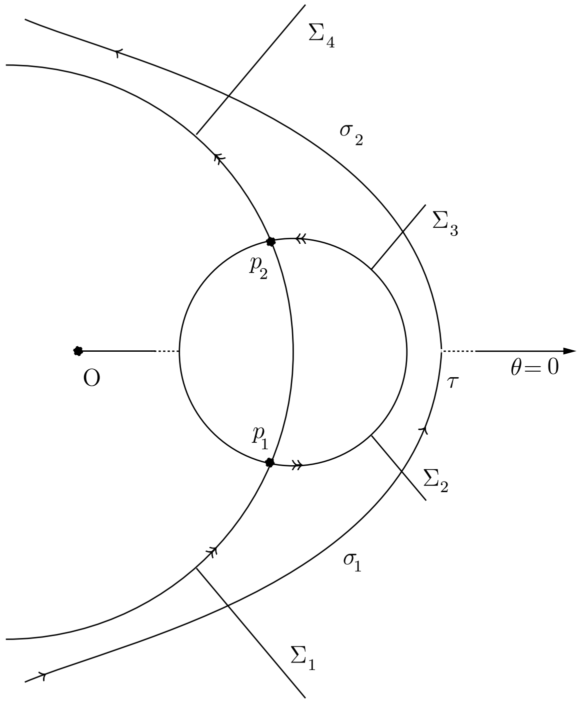

FIG. 1. Blow-up geometry of a characteristic direction of a system in $S_{k,\omega}$.

To compute $\Delta_0^{\bar{\varepsilon}}$ we take the following decomposition:

$$\Delta_0^{\bar{\varepsilon}} = \sigma_2 \circ \tau \circ \sigma_1,$$

where σ_1 is the transition map of the flow in a neighbourhood of p_1, σ_2 is the transition map of the flow in a neighbourhood of p_2 and τ is the transition map of the flow between these neighbourhoods. To compute σ_1 we work in the coordinates of system (3). This map is now the transition between $\Sigma_1 = \{[U, 0] \times \{z = -\varepsilon\}\}$ for $U < 0$ close enough to zero, and $\Sigma_2 = \{\{u = -\delta\} \times [Z, 0)\}$ for $Z < 0$, also close enough to zero.

Again in the coordinates of system (3), σ_2 is the transition between $\Sigma_3 = \{\{u = \delta\} \times (0, \tilde{Z})\}$ for $\tilde{Z} > 0$ small enough, and $\Sigma_4 = \{(0, \tilde{U}] \times \{z = \varepsilon\}\}$ for $\tilde{U} > 0$ small enough.

Lastly, τ is the transition map of the flow from Σ_2 to Σ_3. To compute it, we work in the coordinates of system (4).

We denote

$$\sigma_1(\bar{x}) = a|\bar{x}|^{1/\lambda} + o(|\bar{x}|^{1/\lambda}),$$

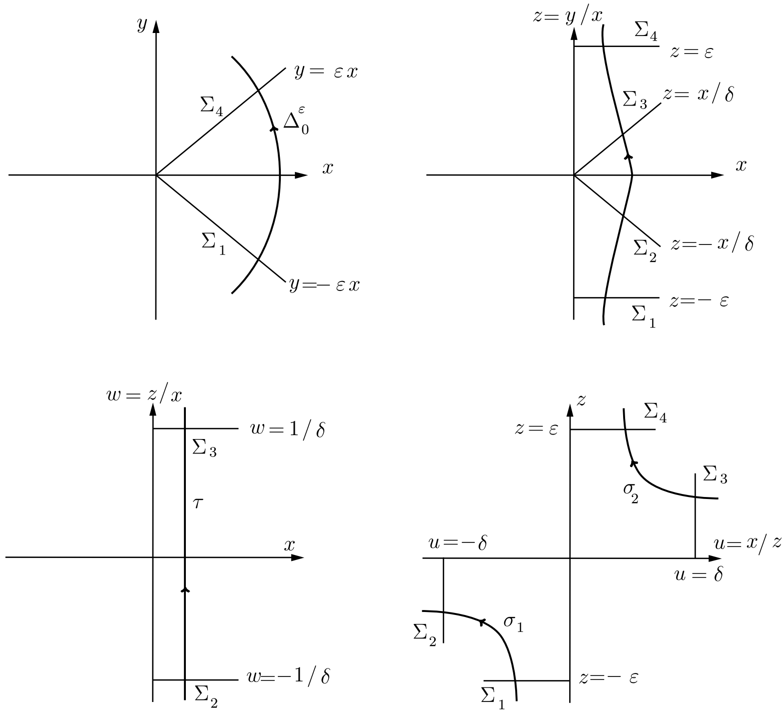

FIG. 2. Blow-up of the characteristic direction $\{\theta = 0\}$ in local coordinates.

$$\sigma_2(\bar{x}) = b\bar{x}^{\lambda} + o(\bar{x}^{\lambda}),$$

$$\tau(\bar{x}) = c\bar{x} + o(\bar{x})$$

and we use in the next computations the following notation: $(a,b)^{(n)}$, where $n \in \{1,2,3\}$, means a point expressed in the original coordinates (x,y) of our system if $n=1$; expressed in coordinates (x,w) of system (4) if $n=2$; and in coordinates (u,z) of system (3) if $n=3$. We get

$$\begin{aligned}\Delta_0^{\bar{\varepsilon}} &= \sigma_2 \circ \tau \circ \sigma_1((u,-\varepsilon)^{(3)}) = \sigma_2 \circ \tau((-\delta, a|u|^{1/\lambda} + o(|u|^{1/\lambda}))^{(3)})\\ &= \sigma_2 \circ \tau\left(\left(-\delta a|u|^{1/\lambda} + o(|u|^{1/\lambda}), -\frac{1}{\delta}\right)^{(2)}\right) = \sigma_2\left(\left(-c\delta a|u|^{1/\lambda} + o(|u|^{1/\lambda}), \frac{1}{\delta}\right)^{(2)}\right)\\ &= \sigma_2((\delta, -ca|u|^{1/\lambda} + o(|u|^{1/\lambda}))^{(3)}) = \sigma_2((bc^{\lambda}(-a)^{\lambda}|u| + o(|u|), \varepsilon)^{(3)}).\end{aligned}$$

Now, since $(u, -\varepsilon)^{(3)} = (-\varepsilon u, \varepsilon^2 u)^{(1)}$, we obtain

$$(bc^{\lambda}(-a)^{\lambda}|u|, \varepsilon)^{(3)} = (\varepsilon bc^{\lambda}(-a)^{\lambda}|u| + o(|u|), \varepsilon^2 bc^{\lambda}(-a)^{\lambda}|u| + o(|u|))^{(1)}.$$

Then for a point $(x, y)^{(1)}$, it is easy to check that

$$\Delta_0^{\bar{\varepsilon}}(r) = bc^{\lambda}(-a)^{\lambda} r + o(r), \tag{8}$$

where $r^2 = x^2 + y^2$. Now we compute a and b. Using Lemma 6, adapted to the corresponding quadrant, we obtain

$$a = \frac{-\varepsilon}{(\delta)^{1/\lambda}} \frac{\exp\{F(-\delta)\}}{\exp\{\frac{1}{\lambda}G(-\varepsilon)\}} \qquad \text{and} \qquad b = \frac{\delta}{\varepsilon^{\lambda}} \frac{\exp\{G(\varepsilon)\}}{\exp\{\lambda F(\delta)\}}.$$

To compute c we consider system (4), and we use the first-order variational equations along the solution $\{x = 0\}$, from $\{w = -1/\delta\}$ to $\{w = 1/\delta\}$, obtaining

$$c = \exp\left\{ \int_{-1/\delta}^{1/\delta} \left(\frac{\partial}{\partial x} \frac{p(x, w)}{q(x, w)} \right)\bigg|_{\{x=0\}} dw \right\} =: \exp\left\{ \int_{-1/\delta}^{1/\delta} h(w)\, dw \right\}.$$

Hence, from Eq. (8), and taking into account the above expressions, we have

$$\Delta_0^{\bar{\varepsilon}}(r) = D_0^{\bar{\varepsilon}} r + o(r),$$

where

$$D_0^{\bar{\varepsilon}} = \exp\left\{ F(\varepsilon) - F(-\varepsilon) - \lambda(G(\delta) - G(-\delta)) + \lambda \int_{-1/\delta}^{1/\delta} h(w)\, dw \right\}.$$

Observe now that D_0^{ε} does not depend on δ, hence

$$A := -\lambda(G(\delta) - G(-\delta)) + \lambda \int_{-1/\delta}^{1/\delta} h(w)\, dw$$

is constant for all $\delta > 0$ small enough. Therefore,

$$A = \lim_{\delta \to 0} A = \lambda\, \mathrm{PV} \quad \int_{-\infty}^{\infty} h(w)\, dw.$$

This ends the proof of the lemma. ∎

Remark 8. Observe that

$$\begin{aligned}\lim_{\varepsilon\to 0} D_0^{\bar{\varepsilon}} &= \lim_{\varepsilon\to 0} \exp\left\{F(\varepsilon) - F(-\varepsilon) + \lambda\,\mathrm{PV} \quad \int_{-\infty}^{\infty} h(w)\,dw\right\}\\ &= \exp\left\{\lambda\,\mathrm{PV} \quad \int_{-\infty}^{\infty} h(w)\,dw\right\}\\ &= \exp\left\{\lambda\,\mathrm{PV} \quad \int_{-\infty}^{\infty}\left(\left(\frac{\partial}{\partial x}\frac{p(x,w)}{q(x,w)}\right)\bigg|_{\{x=0\}}\right) dw\right\}.\end{aligned}$$

Proof of Theorem 2. Let $X \in S_{k.\omega}$ and set $\omega \cup \{\omega + \pi\} = \{\theta_1, \ldots, \theta_l\}$. Observe that it is not restrictive to assume that $\{\theta = 0\}$ is not a characteristic direction. Let $\varepsilon > 0$ be small enough such that $\{(\theta_j - \varepsilon, \theta_j + \varepsilon)\}_{j\in\{1,\ldots,l\}}$ is a collection of disjoint intervals. Set $S_\varepsilon = \bigcup_{j=1}^{l}(\theta_j - \varepsilon, \theta_j + \varepsilon)$ and $I_\varepsilon = \big([0, 2\pi]\backslash S_\varepsilon\big)$.

Taking polar coordinates (r, θ) given by the change $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$; and re-scaling the time by $ds/dt = r^{m-1}$, we obtain the new system

$$\begin{aligned}\dot{r} &= \Re(r,\theta) = R_k(X)(\theta)r + o(r^2),\\ \dot{\theta} &= \Theta(r,\theta) = F_k(X)(\theta) + o(r).\end{aligned} \tag{9}$$

Now, integrating the first-order variational equations of system (9) associated with the orbit $\{r = 0\}$, we have that the transition map T_j^ε from $\{\theta = \theta_j + \varepsilon\}$ to $\{\theta = \theta_{j+1} - \varepsilon\}$ (which is regular) and is given by

$$T_j^\varepsilon(r_0) = \exp\left\{\int_{\theta_j+\varepsilon}^{\theta_{j+1}-\varepsilon} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\,d\theta\right\} r_0 + o(r_0).$$

Also, let T_0^ε be the regular transition map from $\{\theta = 0\}$ to $\{\theta = \theta_1 - \varepsilon\}$, and T_l^ε be the regular transition from $\{\theta = \theta_l + \varepsilon\}$ to $\{\theta = 2\pi\}$. Hence, we get that $\Pi = T_l^\varepsilon \circ \Delta_l^\varepsilon \circ T_{l-1}^\varepsilon \cdots T_2^\varepsilon \circ \Delta_2^\varepsilon \circ T_1^\varepsilon \circ \Delta_1^\varepsilon \circ T_0^\varepsilon$ is a composition of regular and semi-regular maps with nonvanishing linear leading terms, hence we can write

$$\Pi(x) \;=\; V_1\,x \;+\; o(x),$$

where V_1 is the product of the principal terms of the maps Δ_j^ε and δ_j^ε, for $j = 1, \ldots, l$. Therefore, for all $\varepsilon > 0$ small enough:

$$V_1 \;=\; \left(\prod_{j=1}^{l} D_j^\varepsilon\right)\exp\left\{\int_{I_\varepsilon} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\,d\theta\right\}. \tag{10}$$

Observe that the terms depending on ε in the integrals appearing in each D_j^ε (given in Lemma 7) are nonsingular, and that for every supplementary characteristic direction the principal values appearing in the expressions of D_j^ε, which are given in Lemma 7, are the same but with opposite sign, and then they cancel. Therefore $\lim_{\varepsilon\to 0}\prod_{j=1}^{l} D_j^{\varepsilon} = 1$. Taking $\varepsilon \to 0$ in Eq. (10), we have that the GPV exists and

$$V_1 = \exp\left\{\text{GPV} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\, d\theta\right\}. \quad \blacksquare$$

4. PRACTICAL USE OF THEOREMS 1 AND 2

In order to make Theorems 1 and 2, useful we have to consider if it is possible to check if a concrete vector field is an element of $S_{k,\omega}$. In this case, we also study when it is possible to compute analytically

$$\text{GPV} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\, d\theta.$$

The method that we develop shows that this global principal value is always numerically computable. Remember that this value, when it is not zero, gives the stability of the origin.

In this section, we study both problems. The main result (Theorem 13) is stated in Section 4.2. We finish the section giving some concrete examples.

4.1. Semi-algebraic Characterization of $S_{k,\omega}$

In order to check if a vector field $X \in \Phi_k$ belongs to $S_{k,\omega}$ for some ω, conditions (a)–(d) before Theorem 1 have to be checked. In particular, the values θ_j, zeros $F_k(X)(\theta)$ have to be computed. This is not always possible, but taking into account that all its real zeros have to be double zeros, we can compute them for $k \leqslant 7$. We develop this study in the sequel.

First, notice that since $F_k(X)$ is a homogeneous trigonometric polynomial of degree $k+1$, the problem of determining its zeros can be reduced to the determination of the zeros of a polynomial of degree $k+1$ with just one variable,

$$P_{k+1}(\tan(\theta)) = F_k(X)(\theta)/\cos^{k+1}(\theta).$$

Given a real polynomial P_k of degree k, we define the following (finite) sequence of polynomials

$$\begin{aligned} M_0(x) &:= P_k(x),\\ M_l(x) &:= \text{g.c.d. } (M_{l-1}(x), M_{l-1}'(x)), \qquad 1 \leqslant l \leqslant m. \end{aligned} \tag{11}$$

We stop the sequence when some M_l is a nonzero real constant. Remember that given any real polynomial $Q(x)$ it is possible by using the Sturm sequence (see [15, pp. 281–282]) to determine, by means of algebraic inequalities, its number of real roots (NRR(Q)) and its number of complex (nonreal) roots (NCR(Q)) without taking into account their multiplicities. From the two functions NRR and NCR, and from the sequence $M_i(x)$, $i = 0, 1, \ldots, m$ we can prove the following result.

PROPOSITION 9. *Let P_k be a real polynomial of degree k. Associated with it, consider the sequence of polynomials $M_i(x)$, $i = 0, 1, \ldots, m$ given in* (11). *Define also $M_{m+1}(x) \equiv M_{m+2}(x) \equiv 1$. Then, the following statements hold*:

(i) *The number* $\mathrm{NRR}(M_i) - \mathrm{NRR}(M_{i+1})$ *is the number of real roots of P_k that have exactly multiplicity $i + 1$.*

(ii) *The number* $\mathrm{NCR}(M_i) - \mathrm{NCR}(M_{i+1})$ *is the number of complex* (*nonreal*) *roots of P_k that have exactly multiplicity $i + 1$.*

(iii) *If*

$$\deg\left(\frac{M_i(x)M_{i+2}(x)}{M_{i+1}^2(x)}\right) \leqslant 4, \quad i = 0, \ldots, m,$$

then all the roots of $P_k(x)$ can be computed by radicals.

Proof. The proofs of (i) and (ii) follow similar steps. It suffices to recall that a linear or quadratic irreducible factor in $\mathbf{R}[x]$ of a polynomial $Q(x) \in \mathbf{R}[x]$ has multiplicity k if and only if it has multiplicity $k - 1$ in the polynomial g.c.d. $(Q(x), Q'(x))$.

The proof of (iii) follows from the above considerations and the fact that all polynomials of degree at most 4 can be solved by radicals. Observe that any irreducible factor in $\mathbf{R}[x]$ of $M_{i+2}(x)$, with multiplicity $k \geqslant 2$, does not appear as a factor of $M_i(x)M_{i+2}(x)/M_{i+1}^2(x)$. ∎

COROLLARY 10. *By using the same notations as that in Proposition* 9. *The following results hold*:

(i) *P_k does not change sign if and only if*

$$\mathrm{NRR}(M_{2i}) = \mathrm{NRR}(M_{2i+1}), \qquad i = 0, 1, \ldots, [m/2].$$

(ii) *P_k has all its real roots of multiplicity two if and only if*

$$\mathrm{NRR}(M_0) = \mathrm{NRR}(M_1) \qquad \text{and} \qquad \mathrm{NRR}(M_2) = 0.$$

(iii) *If $k = 2, 4, 6$ or 8 and P_k has its real roots of multiplicity two, then all these real roots can be computed by using radicals.*

Note that from Theorem 1, statement (i) of the above corollary gives a necessary condition—which can be checked algebraically—for an element

$X \in \Phi_k$ to be monodromic and to be in some $\Phi_{k,w}$. In a similar way, statement (ii) gives a necessary condition for an element of Φ_k to be in $S_{k,w}$ for some w. Furthermore, in the following section we will see—as a consequence of statement (iii) and some technical results—when is a semi-algebraic problem to decide if a vector field $X \in \Phi_k$, also belongs to $S_{k,w}$. Finally, we also study when GPV $\int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta$ is algebraically computable.

4.2. *On the Computation of Global Principal Values*

All vector fields in $S_{k,w}$ satisfy, among other things, that

$$F_k(X)(\theta_j) = F_k(X)'(\theta_j) = R_k(\theta_j) = 0 \qquad \text{and} \qquad F_k(X)''(\theta_j) \neq 0$$
$$\text{for all } \theta_j \in w. \tag{12}$$

In the next lemma we will prove that GPV $\int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta$ exists for the subset of vector fields in Φ_k satisfying (12). Observe that this subset contains $S_{k,w}$. We also give a method to reduce its computation to the computation of a nonsingular integral.

LEMMA 11. *Let $R(\theta)$ and $F(\theta)$ be homogeneous trigonometric polynomials of degree k satisfying that*

$$R(\theta) = \Pi(\theta) r(\theta) \quad \textit{and} \quad F(\theta) = \Pi^2(\theta) f(\theta)$$

with $f(\theta) > 0$ and $\Pi(\theta) = \prod_{i=1}^{l} \sin(\theta - \theta_i)$.

Then GPV $\int_0^{2\pi} \frac{R(\theta)}{F(\theta)} d\theta$ *exists, and can be computed as*

$$\int_0^{2\pi} \frac{s(\theta)}{f(\theta)} d\theta,$$

where $s(\theta)$ is the homogeneous trigonometric polynomial of degree $k - 2l$ given by

$$s(\theta) := \frac{r(\theta) - \Pi(\theta) f(\theta) \left(\sum_{i=1}^{l} A_i \cot(\theta - \theta_i) \right)}{\Pi(\theta)}$$

and $A_i = \frac{r(\theta_i)}{\Pi'(\theta_i) f(\theta_i)}$, $i = 1, 2, \ldots, l$.

Proof. Direct computations show that

$$\frac{r(\theta)}{\Pi(\theta) f(\theta)} = \sum_{i=1}^{l} \frac{A_i \cos(\theta - \theta_i)}{\sin(\theta - \theta_i)} + \frac{s(\theta)}{f(\theta)}.$$

The result follows from the fact that

$$\text{GPV} \quad \int_0^{2\pi} \frac{\cos(\theta-\theta_i)}{\sin(\theta-\theta_i)}\, d\theta = 0$$

and

$$\text{GPV} \quad \int_0^{2\pi} \frac{s(\theta)}{f(\theta)}\, d\theta = \int_0^{2\pi} \frac{s(\theta)}{f(\theta)}\, d\theta,$$

because $f(\theta)$ is always positive. ∎

Remark 12. Remember that by using the transformation $z = e^{i\theta}$, if we write $\frac{s(\theta)}{f(\theta)} = q(\cos\theta, \sin\theta)$, then

$$\int_0^{2\pi} \frac{s(\theta)}{f(\theta)}\, d\theta = \int_{|z|=1} \frac{1}{iz} q\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)$$

and this integral can be studied by using Cauchy's Residues Theorem.

Finally, as a consequence of statement (iii) of Corollary 10, Lemma 11 and Remark 12 we have the following result:

THEOREM 13. *Consider a vector field $X \in \Phi_k$. Then it is a semi–algebraic problem to determine if there exists w such that $X \in S_{k,w}$ when $k = 1, 3, 5$ or 7. Furthermore, if $(k, w) \notin \{(5, \emptyset), (7, \emptyset), (7, \{\theta\})\}$, the global principal value* GPV $\int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)}\, d\theta$ *can be algebraically computed.*

Remark 14. (i) Observe that the stability problem in the case that $(k, w) \in \{(5, \emptyset), (7, \emptyset), (7, \{\theta\})\}$, cannot be algebraically solved because in this situation the rational integral given in Remark 12 cannot be computed in general.

(ii) There are a lot of cases in which the monodromy problem can be solved even for $k > 7$. The computation of the global principal value also depends on the computation of the rational integral given in Remark 12.

4.3. Concrete Examples

In order to give concrete examples of application of our results we study in this section the case $k = 3$.

From Theorem 1 and results of Section 4.1, if the origin of a vector field $X \in \Phi_3$ is monodromic, then there exists a linear change of coordinates such that $F_3(X)(\theta)$ has one of the following expressions:

(1) $F_3(X)(\theta) = \alpha \sin^4(\theta)$,

(2) $F_3(X)(\theta) = \alpha \sin^2(\theta) \cos^2(\theta)$,

(3) $F_3(X)(\theta) = \sin^2(\theta)S_2(\cos(\theta), \sin(\theta))$,

(4) $F_3(X)(\theta) = S_4(\cos(\theta), \sin(\theta))$,

where $\alpha \in \mathbf{R}\setminus\{0\}$, and S_l are homogeneous polynomials of degree l which do not vanish for $\theta \in [0, 2\pi)$.

Case (1) is not included in our study (it corresponds with a zero of multiplicity four). Case (4) corresponds to the case of nonexistence of characteristic directions and it is well known that the Poincaré–Lyapunov theory works in this situation. Cases (2) and (3) are the ones for which our results apply. Taking into account that any vector field X having a monodromic singular point at the origin must satisfy (12), we have that cases (2) and (3) corresponds with vector field X, of the form

$$X_3(x, y) = (Bx^2y, Gxy^2) \quad \text{with } G > B, \tag{13}$$

or

$$X_3(x, y) = (Bx^2y + Cxy^2 + Dy^3, Gxy^2 + Hy^3)$$

$$\text{with } (H - C)^2 + 4D(G - B) < 0 \quad \text{and} \quad D < 0. \tag{14}$$

For system (13), Lemma 11 gives that GPV $\int_0^{2\pi} \frac{R_3(X)(\theta)}{F_3(X)(\theta)}\, d\theta = 0$. While for system (14) gives that

$$\text{GPV} \quad \int_0^{2\pi} \frac{R_3(X)(\theta)}{F_3(X)(\theta)}\, d\theta = \mathscr{I}\begin{pmatrix} \frac{CG-BH}{G-B} & \frac{DG+G^2-GB}{2(G-B)} & H \\ G - B & \dfrac{H - C}{2} & -D \end{pmatrix}, \tag{15}$$

where

$$\mathscr{I}\begin{pmatrix} P & Q & R \\ p & q & r \end{pmatrix} := \int_0^{2\pi} \frac{P\cos^2(\theta) + 2Q\sin(\theta)\cos(\theta) + R\sin^2(\theta)}{p\cos^2(\theta) + 2q\sin(\theta)\cos(\theta) + r\sin^2(\theta)}\, d\theta$$
$$= \frac{2\pi}{4q^2 + (p - r)^2}\left(4Qq + (P - R)(p - r) + \frac{[2(P + R)q^2 - 2Qq(p + r) + (pR - Pr)(p - r)]}{\sqrt{pr - q^2}}\right),$$

for $q^2 - pr < 0$ and $4q^2 + (p - r)^2 \neq 0$, or

$$\mathscr{I}\begin{pmatrix} P & Q & R \\ p & q & r \end{pmatrix} := \frac{\pi(P + R)}{p}$$

if $4q^2 + (p-r)^2 = 0$. To get the above expression we have used the formulas that appear in pp. 150–151 of [8].

Note that to ensure that the global principal values computed above give the stability of a vector field $X \in \Phi_3$, with X_3 of the prescribed form we also have to check conditions (c) and (d) of the definition of $S_{3,w}$.

We end this section with the following example. Consider the system

$$\begin{aligned} \dot{x} &= bx^2y + axy^2 - by^3 - x^4, \\ \dot{y} &= 4bxy^2 + ay^3 + 2x^5. \end{aligned} \tag{16}$$

It is easy to check that $X \in S_{3,\{0\}}$ if and only if $b > 1/4$. Furthermore, for any nonzero value of b

$$\text{GPV} \quad \int_0^{2\pi} \frac{R_3(X)(\theta)}{F_3(X)(\theta)} d\theta = \mathscr{I}\begin{pmatrix} a & \frac{4}{3}b & a \\ 3b & 0 & b \end{pmatrix} = \frac{2\pi\sqrt{3}a}{3b}.$$

Therefore if $b > 1/4$, the origin of (16) is monodromic. Furthermore, it is an attractor (respectively, a repellor) if $a < 0$ (respectively, $a > 0$). Observe that if $a = 0$ it is easy to see that the origin is a reversible centre. If $b = 1/4$, the critical point that appears on $\{x = 0\}$ for system (4) is elementary degenerate. If $b < 1/4$ Theorem 1 assures that $X \notin N_{3,\{0\}}$. Therefore, if $b \leqslant 1/4$ then the origin is not monodromic.

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